Conditional Inference in Subjective Logic

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Abstract – *The interpretations of conditionals and conditional inference are often disputed. The classic logic conditional called material implication can easily be proven to be invalid, and the conditional inference rule Modus Ponens represents a tautology that becomes invalid in the face of contrary evidence from realistic scenarios. The foundations of conditionals and conditional inference seem plagued with problems and also seem unable to realistically model causal relationships in the world around us. Now introduce the concepts of ignorance and uncertainty into the framework and it seems to get even fuzzier because the traditional tools of logic or probabilistic conditional inference can no longer be applied. This paper introduces a conditional inference operator that explicitly incorporates ignorance and uncertainty, thereby making it suitable in situations of partial ignorance and imperfect information.*

Keywords: Modus Ponens, Implication, Causality, Belief theory, Conditional inference, Subjective logic, Probability

1 Introduction

Conditionals are propositions like *"If the reserve bank does not reduce the interest rate, the recession will continue"* or *"If it rains, Michael will carry an umbrella"* which are of the form "IF x THEN y" where x marks the antecedent and y the consequent. An equivalent way of expressing conditionals is through the concept of implication, so that *"If it rains, Michael will carry an umbrella"* is equivalent to *"The fact that it rains implies that Michael carries an umbrella"*.

When making assertions of conditionals with antecedent and consequent, which can be evaluated as TRUE or FALSE propositions, we are in fact evaluating a proposition which can itself be considered TRUE or FALSE. A conditional is of course not always true, and it is quite common to hear utterings like: *"I don't believe that the recession will continue if the reserve bank does not reduce the interest rate"* or *"Is it really true that Michael will carry an umbrella if it rains?"* which are questioning the truth of the conditionals.

The importance of conditionals is evidenced by the fact that both logic and probability calculus have mechanisms for handling the evaluation of conditionals. In logic, Modus Ponens (MP) is the tool of choice. It is used in any field of logic that requires deduction to take place. In probability calculus, Bayes rule for conditional evaluation is the tool of choice. Both frameworks exclude one important ingredient. The treatment of uncertainty. As real systems are normally riddled with uncertainty, neither of the above mentioned frameworks can be effectively used in real systems. Thus, there is a need for an uncertainty framework with facilities for reasoning about conditionals.

Subjective logic[4] is a logic of uncertain beliefs about propositions, is related to belief theory, and is compatible with binary logic and probability calculus. Subjective logic contains operators that correspond to standard logic 'AND', 'OR' and 'NOT' as well as the non-standard operators 'consensus' and 'discounting'. An online demonstration of these operators can be found at [2].

This paper describes a new operator called *conditional inference* and highlights the usefulness of subjective logic over binary logic and probability calculus because it is possible to model situations where the antecedent, the consequent and the conditional itself are uncertain. Section 2 details our representation of uncertain beliefs, while section 3 discusses the belief metric called *opinion* which is used for representing beliefs about propositions. Section 4 describes the conditional inference operator of subjective logic, and section 5 describes examples that show how the conditional inference operator can be applied. Section 6 provides a discussion on the confusion surrounding the incarnations of conditional inference in standard logic and probability calculus. Section 7 summarises the contribution of this paper.

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2 Representing Uncertain Beliefs

The first step in applying the Dempster-Shafer belief model [8] is to define a set of possible states of a given system, called the *frame of discernment* denoted by Θ. The states in Θ are assumed to be exhaustive and mutually exclusive, and will therefore be called atomic states.

The powerset of Θ, denoted by 2^Θ, contains all possible unions of the atomic states in Θ including Θ itself. It is assumed that only one atomic state can be true at any one time. If a state is assumed to be true, then all superstates are considered true as well.

An observer who believes that one or several states in the powerset of Θ might be true can assign belief masses to these states. Belief mass on an atomic state $x \in 2^\Theta$ is interpreted as the belief that the state in question is true. Belief mass on a non-atomic state $x \in 2^\Theta$ is interpreted as the belief that one of the atomic states it contains is true, but that the observer is uncertain about which of them is true.

In general, a belief mass assignment (BMA) denoted by m is defined as a function from 2^Θ to $[0,1]$ satisfying:

$$\sum_{x \subseteq \Theta \setminus \emptyset} m(x) = 1. \tag{1}$$

Each subset $x \in 2^\Theta$ such that $m(x) > 0$ is called a *focal element* of m. A BMA where $m(\Theta) = 0$ is called *dogmatic*. Given a particular frame of discernment and a BMA, the Dempster-Shafer thery [8] defines a belief function $b(x)$. In addition, subjective logic [4] defines a disbelief function $d(x)$, an uncertainty function $u(x)$, a relative atomicity function $a(x/y)$ and a probability expectation $\mathrm{E}(x)$. These are all defined as follows:

$$b(x) \triangleq \sum_{\emptyset \neq y \subseteq x} m(y) \qquad \forall\, x \in 2^\Theta\ , \tag{2}$$

$$d(x) \triangleq \sum_{y \cap x = \emptyset} m(y) \qquad \forall\, x \in 2^\Theta\ , \tag{3}$$

$$u(x) \triangleq \sum_{\substack{y \cap x \neq \emptyset \\ y \not\subseteq x}} m(y) \qquad \forall\, x \in 2^\Theta\ , \tag{4}$$

$$a(x/y) \triangleq \frac{|x \cap y|}{y} \qquad \forall\, x \in 2^\Theta\ , \tag{5}$$

$$\mathrm{E}(x) \triangleq \sum_{y \subseteq \Theta} m_\Theta(y)\ a(x/y) \qquad \forall\, x \in 2^\Theta\ . \tag{6}$$

The relative atomicity function of a subset x relative to the frame of discernment Θ is simply dentoted by $a(x)$.

Subjective logic applies to binary frames of discernment, so in case a frame is larger than binary, a coarsening is required to reduce its size to binary. Coarsening focuses on a particular subset $x \subset \Theta$, and produces a binary frame of discernment X containing x and its complement $\overline{x}$. The powerset of X is $2^X = \{x, \overline{x}, X\}$ which has $2^{|X|} - 1 = 3$ elements when excluding $\emptyset$. We will first describe *simple coarsening* which is described in [4] and subsequently describe *normal coarsening* which has not been described elsewhere. Let the coarsened frame of discernment be $X = \{x, \overline{x}\}$ where $\overline{x}$ is the complement of x in Θ. We will denote by b_x, d_x, u_x and a_x the belief, disbelief, uncertainty and relative atomicity functions of x on X.

According to simple coarsening, these functions are defined as:

$$b_x \triangleq b(x)\ , \tag{7}$$

$$d_x \triangleq d(x)\ , \tag{8}$$

$$u_x \triangleq u(x)\ , \tag{9}$$

$$a_x \triangleq [\mathrm{E}(x) - b(x)]/u(x)\ . \tag{10}$$

This coarsening is called "simple" because the belief, disbelief and uncertainty functions are identical to the original functions on Θ. The simple relative atomicity function on the other hand produces a synthetic relative atomicity value which does not represent the real relative atomicity of x on Θ in general.

Next, the normal coarsening method is described. According to normal coarsening, the belief, disbelief, uncertainty and relative atomicity functions are defined as:

For $\mathrm{E}(x) \geq b(x) + a(x)u(x)$:

$$b_x \triangleq b(x) + (\mathrm{E}(x) - b(x) - a(x)u(x))/(1 - a(x))\ , \tag{11}$$

$$d_x \triangleq d(x)\ , \tag{12}$$

$$u_x \triangleq u(x) - (\mathrm{E}(x) - b(x) - a(x)u(x))/(1 - a(x))\ , \tag{13}$$

$$a_x \triangleq a(x)\ . \tag{14}$$

(15)

For $\mathrm{E}(x) < b(x) + a(x)u(x)$:

$$b_x \triangleq b(x)\ , \tag{16}$$

$$d_x \triangleq d(x) + (b(x) + a(x)u(x) - \mathrm{E}(x))/a(x)\ , \tag{17}$$

$$u_x \triangleq u(x) - (b(x) + a(x)u(x) - \mathrm{E}(x))/a(x)\ , \tag{18}$$

$$a_x \triangleq a(x)\ . \tag{19}$$

This coarsening is called "normal" because the relative atomicity function represents the actual relative atomicity of x on Θ. The relative cardinality of an element in a binary frame of discernment will always be 2, whereas the normal relative atomicity reflects the true relative atomicity of an element relative to the original frame of discernment.

The belief, disbelief and uncertainty functions on X for normal coarsening are in general different from the belief, disbelief and uncertainty functions on Θ so that $b(x) \leq b_x$, $d(x) \leq d_x$, and $u(x) \geq u_x$. The interpretation of the tendency of normal coarsening to decrease the uncertainty and increase the belief and disbelief functions is that belief mass that contribute to the uncertainty function on Θ can have a varying character of uncertainty . When considering for example the frame of discernment $\Theta = \{x_1, x_2, x_3\}$ and focusing on the state $x_1 \cup x_2$, then the belief mass $m(x_2 \cup x_3)$ has less character of uncertainty and should therefore contribute less to the uncertainty function $u_{x_1 \cup x_2}$ than the belief mass $m(\Theta)$.

3 The Opinion Space

For the purpose of having a simple and intuitive representation of uncertain beliefs we use a 3-dimensional metric called *opinion* but which will contain a 4th redundant parameter in order to allow a more compact definition of the conditional inference operator.

It is assumed that all beliefs are held by individuals and the notation will therefore include belief ownership. Let for example agent A express his or her beliefs about the truth of state x in some frame of discernment. We will denote A's belief, disbelief, uncertainty and relative atomicity functions as b_x^A, d_x^A, u_x^A and a_x^A respectively, where the superscript indicates belief ownership and the subscript indicates the belief target.

Definition 1 (Opinion) *Let Θ be a binary frame of discernment containing states x and $\overline{x}$, and let m_Θ be the BMA on Θ held by A where b_x^A, d_x^A and u_x^A represent A's belief, disbelief and uncertainty functions on x in 2^Θ respectively, and let a_x^A represent the relative atomicity of x in Θ. Then A's* opinion *about x, denoted by ω_x^A, is the tuple:*

$$\omega_x^A \triangleq (b_x^A,\ d_x^A,\ u_x^A,\ a_x^A)\ .$$

The three components (b_x, d_x, u_x) satisfy

$$b_x + d_x + u_x = 1 \tag{20}$$

so that one is redundant. As such they represent nothing more than the traditional *Bel* (Belief) and *Pl* (Plausibility) pair of Shaferian belief theory, where *Bel* $= b$ and *Pl* $= b + u$. However, using (*Bel*, *Pl*) instead of $(b,\ d,\ u)$ would have produced unnecessary complexity in the definition of the operators in subjective logic. The probability expectation of an opinion is expressed as:

$$\mathrm{E}(\omega_x) = b_x + a_x u_x \tag{21}$$

It can be shown that $\mathrm{E}(\omega_x) = \mathrm{E}(x)$ holds for both simple and normal coarsening. Eq.(20) defines a triangle that can be used to graphically illustrate opinions as shown in Fig.1.

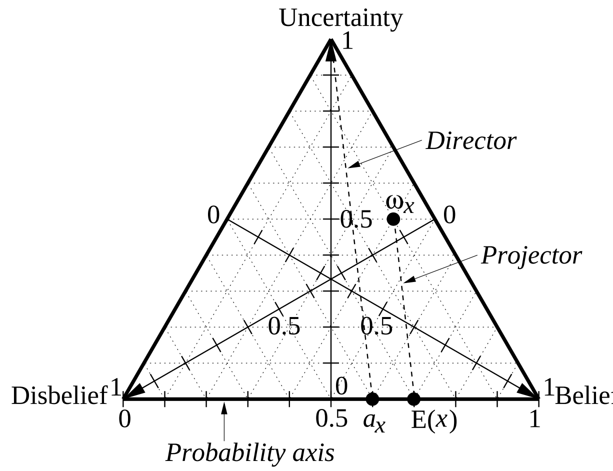

Figure 1: Opinion triangle with ω_x as example

As an example the position of the opinion $\omega_x = (0.40,\ 0.10,\ 0.50,\ 0.60)$ is indicated as a point in the triangle. The horizontal base line between the belief and disbelief corners is called the *probability axis*. As shown in the figure, the probability expectation value $\mathrm{E}(x) = 0.7$ and the relative atomicity $a(x) = 0.60$ can be graphically represented as points on the probability axis. The line joining the top corner of the triangle and the relative atomicity point is called the *director*. The *projector* is parallel to the director and passes through the opinion point ω_x. Its intersection with the probability axis defines the probability expectation value which otherwise can be computed by the formula of Eq.(6). Opinions situated on the probability axis are called *dogmatic opinions*, representing traditional probabilities without uncertainty. The distance between an opinion point and the probability axis can be interpreted as the degree of uncertainty. Opinions situated in the left or right corner, i.e. with either $b = 1$ or $d = 1$ are called *absolute opinions*, corresponding to TRUE or FALSE states in binary logic.

Opinions have an equivalent represention as beta probability density functions (pdf) denoted by $\mathrm{beta}\,(\alpha, \beta)$ through the following bijective mapping:

$$\begin{aligned} &(b_x, d_x, u_x, a_x) \longleftrightarrow \\ &\mathrm{beta}\left(\tfrac{2b_x}{u_x} + 2a_x,\ \ \tfrac{2d_x}{u_x} + 2(1 - a_x)\right). \end{aligned} \tag{22}$$

This means for example that an opinion with $u_x = 1$ and $a_x = 0.5$ which maps to $\mathrm{beta}\,(1, 1)$ is equivalent to a uniform pdf. It also means that a dogmatic opinion with $u_x = 0$ which maps to $\mathrm{beta}\,(b_x\eta,\ d_x\eta)$ where $\eta \to \infty$ is equivalent to a spike pdf with infinitesimal width and infinite height. Dogmatic opinions can thus be interpreted as being based on an infinite amount of evidence.

4 Conditional Inference

In this section we describe the conditional inference operator for subjective logic. It will become evident that MP in binary logic and probability assessment using conditional probabilities are a special case of this operator.

The problem with the interpretation of conditional propositions like 'IF x THEN y' is that when the antecedent is false it is impossible to assert the truth value of the conditional. To resolve this issue, classic logic stipulates that any conditional with a false antecedent must evaluate to true, i.e. the conditional is seen as truth-functional according to the truth table of material implication. However, this position is untenable in practice because there are examples of false conditionals with false antecedent and true consequent. As a result modern text books state for example that for material implication *"the best thing is to take its truth table as defining its intended meaning"* [10] p.263, which basically says that it is a purely theoretical concept with no practical meaning. This will be discussed in section 6 below. What is needed is a complementary conditional that covers the case when the antecedent is false. One that will do the job is the conditional 'IF NOT x THEN y'. Now, if the antecedent x is false, it is possible to determine the validity of 'IF NOT x THEN y'.

Each conditional now provides a part of the picture and can therefore be called sub-conditionals. Together these sub-conditionals form a *complete conditional expression* that provides a complete description of the connection between the antecedent and the consequent. Complete conditional expressions have a two-dimensional truth value because they consist of two sub-conditionals that both have their own truth value.

We adopt the notation $y|x$ to express the sub-conditional 'IF x THEN y', (this in accordance with Stalnaker's [9] assumption that the probability of the proposition x implies y is equal to the probability of y given x) and $y|\overline{x}$ to express the sub-conditional 'IF NOT x THEN y' and assume that it is meaningful to assign opinions (including probabilities) to these sub-conditionals. We also assume that the belief in the truth of the antecedent x and the consequent y can be expressed as opinions. We can then state the following definition.

Definition 2 (Conditional Inference) *Let $\Theta_X = \{x, \overline{x}\}$ and $\Theta_Y = \{y, \overline{y}\}$ be two frames of discernment with arbitrary mutual dependence. Let $\omega_x = (b_x, d_x, u_x, a_x)$, $\omega_{y|x} = (b_{y|x}, d_{y|x}, u_{y|x}, a_{y|x})$ and $\omega_{y|\overline{x}} = (b_{y|\overline{x}}, d_{y|\overline{x}}, u_{y|\overline{x}}, a_{y|\overline{x}})$ be an agent's respective opinions about x being true, about y being true given that x is true and about y being true given that x is false. Let $\omega_{y\|x} = (b_{y\|x}, d_{y\|x}, u_{y\|x}, a_{y\|x})$ be the opinion about y such that:*

Case I: $((b_{y|x} \geq b_{y|\overline{x}}) \wedge (d_{y|x} \geq d_{y|\overline{x}})) \vee ((b_{y|x} \leq b_{y|\overline{x}}) \wedge (d_{y|x} \leq d_{y|\overline{x}}))$

$$b^{\mathrm{I}}_{y\|x} \triangleq b_x b_{y|x} + d_x b_{y|\overline{x}} + u_x(b_{y|x}a_x + b_{y|\overline{x}}(1-a_x))$$

$$d^{\mathrm{I}}_{y\|x} \triangleq b_x d_{y|x} + d_x d_{y|\overline{x}} + u_x(d_{y|x}a_x + d_{y|\overline{x}}(1-a_x))$$

$$u^{\mathrm{I}}_{y\|x} \triangleq b_x u_{y|x} + d_x u_{y|\overline{x}} + u_x(u_{y|x}a_x + u_{y|\overline{x}}(1-a_x))$$

$$a^{\mathrm{I}}_{y\|x} \triangleq a_y \ .$$

Case II.A.1: $((b_{y|x} > b_{y|\overline{x}}) \wedge (d_{y|x} < d_{y|\overline{x}})) \wedge (\mathrm{E}(\omega_{y\|x}) \leq b_{y|\overline{x}} + a_y(1 - b_{y|\overline{x}} - d_{y|x})) \wedge (b_x + a_x u_x \leq a_x)$

$$b^{\mathrm{II.A.1}}_{y\|x} \triangleq b^{\mathrm{I}}_{y\|x} - \frac{u_x a_x a_y (b^{\mathrm{I}}_{y\|x} - b_{y|\overline{x}})}{(b_x + a_x u_x) a_y}$$

$$d^{\mathrm{II.A.1}}_{y\|x} \triangleq d^{\mathrm{I}}_{y\|x} - \frac{u_x a_x (1-a_y)(b^{\mathrm{I}}_{y\|x} - b_{y|\overline{x}})}{(b_x + a_x u_x) a_y}$$

$$u^{\mathrm{II.A.1}}_{y\|x} \triangleq u^{\mathrm{I}}_{y\|x} + \frac{u_x a_x (b^{\mathrm{I}}_{y\|x} - b_{y|\overline{x}})}{(b_x + a_x u_x) a_y}$$

$$a^{\mathrm{II.A.1}}_{y\|x} \triangleq a_y \ .$$

Case II.A.2: $((b_{y|x} > b_{y|\overline{x}}) \wedge (d_{y|x} < d_{y|\overline{x}})) \wedge (\mathrm{E}(\omega_{y\|x}) \leq b_{y|\overline{x}} + a_y(1 - b_{y|\overline{x}} - d_{y|x})) \wedge (b_x + a_x u_x > a_x)$

$$b^{\mathrm{II.A.2}}_{y\|x} \triangleq b^{\mathrm{I}}_{y\|x} - \frac{u_x (1-a_x) a_y (b^{\mathrm{I}}_{y\|x} - b_{y|\overline{x}})}{(d_x + (1-a_x) u_x) a_y}$$

$$d^{\mathrm{II.A.2}}_{y\|x} \triangleq d^{\mathrm{I}}_{y\|x} - \frac{u_x (1-a_x)(1-a_y)(b^{\mathrm{I}}_{y\|x} - b_{y|\overline{x}})}{(d_x + (1-a_x) u_x) a_y}$$

$$u^{\mathrm{II.A.2}}_{y\|x} \triangleq u^{\mathrm{I}}_{y\|x} + \frac{u_x (1-a_x)(b^{\mathrm{I}}_{y\|x} - b_{y|\overline{x}})}{(d_x + (1-a_x) u_x) a_y}$$

$$a^{\mathrm{II.A.2}}_{y\|x} \triangleq a_y \ .$$

Case II.B.1: $((b_{y|x} > b_{y|\overline{x}}) \wedge (d_{y|x} < d_{y|\overline{x}})) \wedge (\mathrm{E}(\omega_{y\|x}) > b_{y|\overline{x}} + a_y(1 - b_{y|\overline{x}} - d_{y|x})) \wedge (b_x + a_x u_x \leq a_x)$

$$b^{\mathrm{II.B.1}}_{y\|x} \triangleq b^{\mathrm{I}}_{y\|x} - \frac{u_x a_x a_y (d^{\mathrm{I}}_{y\|x} - d_{y|x})}{(b_x + a_x u_x)(1-a_y)}$$

$$d^{\mathrm{II.B.1}}_{y\|x} \triangleq d^{\mathrm{I}}_{y\|x} - \frac{u_x a_x (1-a_y)(d^{\mathrm{I}}_{y\|x} - d_{y|x})}{(b_x + a_x u_x)(1-a_y)}$$

$$u^{\mathrm{II.B.1}}_{y\|x} \triangleq u^{\mathrm{I}}_{y\|x} + \frac{u_x a_x (d^{\mathrm{I}}_{y\|x} - d_{y|x})}{(b_x + a_x u_x)(1-a_y)}$$

$$a^{\mathrm{II.B.1}}_{y\|x} \triangleq a_y \ .$$

Case II.B.2: $((b_{y|x} > b_{y|\overline{x}}) \wedge (d_{y|x} < d_{y|\overline{x}})) \wedge$
$(\mathrm{E}(\omega_{y\|x}) > b_{y|\overline{x}} + a_y(1 - b_{y|\overline{x}} - d_{y|x}))$
$\wedge\ (b_x + a_x u_x > a_x)$

$$b^{\mathrm{II.B.2}}_{y\|x} \triangleq b^{\mathrm{I}}_{y\|x} - \frac{u_x(1-a_x)a_y(d^{\mathrm{I}}_{y\|x} - d_{y|x})}{(d_x+(1-a_x)u_x)(1-a_y)}$$

$$d^{\mathrm{II.B.2}}_{y\|x} \triangleq d^{\mathrm{I}}_{y\|x} - \frac{u_x(1-a_x)(1-a_y)(d^{\mathrm{I}}_{y\|x} - d_{y|x})}{(d_x+(1-a_x)u_x)(1-a_y)}$$

$$u^{\mathrm{II.B.2}}_{y\|x} \triangleq u^{\mathrm{I}}_{y\|x} + \frac{u_x(1-a_x)(d^{\mathrm{I}}_{y\|x} - d_{y|x})}{(d_x+(1-a_x)u_x)(1-a_y)}$$

$$a^{\mathrm{II.B.2}}_{y\|x} \triangleq a_y \ .$$

Case III.A.1: $(b_{y|x} < b_{y|\overline{x}}) \wedge (d_{y|x} > d_{y|\overline{x}}) \wedge$
$(\mathrm{E}(\omega_{y\|x}) \le b_{y|x} + a_y(1 - b_{y|x} - d_{y|\overline{x}}))$
$\wedge\ (b_x + a_x u_x \le a_x)$

$$b^{\mathrm{III.A.1}}_{y\|x} \triangleq b^{\mathrm{I}}_{y\|x} - \frac{u_x a_x a_y(b^{\mathrm{I}}_{y\|x} - b_{y|x})}{(b_x+a_x u_x)a_y}$$

$$d^{\mathrm{III.A.1}}_{y\|x} \triangleq d^{\mathrm{I}}_{y\|x} - \frac{u_x a_x (1-a_y)(b^{\mathrm{I}}_{y\|x} - b_{y|x})}{(b_x+a_x u_x)a_y}$$

$$u^{\mathrm{III.A.1}}_{y\|x} \triangleq u^{\mathrm{I}}_{y\|x} + \frac{u_x a_x(b^{\mathrm{I}}_{y\|x} - b_{y|x})}{(b_x+a_x u_x)a_y}$$

$$a^{\mathrm{III.A.1}}_{y\|x} \triangleq a_y \ .$$

Case III.A.2: $(b_{y|x} < b_{y|\overline{x}}) \wedge (d_{y|x} > d_{y|\overline{x}}) \wedge$
$(\mathrm{E}(\omega_{y\|x}) \le b_{y|x} + a_y(1 - b_{y|x} - d_{y|\overline{x}}))$
$\wedge\ (b_x + a_x u_x > a_x)$

$$b^{\mathrm{III.A.2}}_{y\|x} \triangleq b^{\mathrm{I}}_{y\|x} - \frac{u_x(1-a_x)a_y(b^{\mathrm{I}}_{y\|x} - b_{y|x})}{(d_x+(1-a_x)u_x)a_y}$$

$$d^{\mathrm{III.A.2}}_{y\|x} \triangleq d^{\mathrm{I}}_{y\|x} - \frac{u_x(1-a_x)(1-a_y)(b^{\mathrm{I}}_{y\|x} - b_{y|x})}{(d_x+(1-a_x)u_x)a_y}$$

$$u^{\mathrm{III.A.2}}_{y\|x} \triangleq u^{\mathrm{I}}_{y\|x} + \frac{u_x(1-a_x)(b^{\mathrm{I}}_{y\|x} - b_{y|x})}{(d_x+(1-a_x)u_x)a_y}$$

$$a^{\mathrm{III.A.2}}_{y\|x} \triangleq a_y \ .$$

Case III.B.1: $(b_{y|x} < b_{y|\overline{x}}) \wedge (d_{y|x} > d_{y|\overline{x}}) \wedge$
$(\mathrm{E}(\omega_{y\|x}) > b_{y|x} + a_y(1 - b_{y|x} - d_{y|\overline{x}}))$
$\wedge\ (b_x + a_x u_x \le a_x)$

$$b^{\mathrm{III.B.1}}_{y\|x} \triangleq b^{\mathrm{I}}_{y\|x} - \frac{u_x a_x a_y(d^{\mathrm{I}}_{y\|x} - d_{y|\overline{x}})}{(b_x+a_x u_x)(1-a_y)}$$

$$d^{\mathrm{III.B.1}}_{y\|x} \triangleq d^{\mathrm{I}}_{y\|x} - \frac{u_x a_x(1-a_y)(d^{\mathrm{I}}_{y\|x} - d_{y|\overline{x}})}{(b_x+a_x u_x)(1-a_y)}$$

$$u^{\mathrm{III.B.1}}_{y\|x} \triangleq u^{\mathrm{I}}_{y\|x} + \frac{u_x a_x(d^{\mathrm{I}}_{y\|x} - d_{y|\overline{x}})}{(b_x+a_x u_x)(1-a_y)}$$

$$a^{\mathrm{III.B.1}}_{y\|x} \triangleq a_y \ .$$

Case III.B.2: $(b_{y|x} < b_{y|\overline{x}}) \wedge (d_{y|x} > d_{y|\overline{x}}) \wedge$
$(\mathrm{E}(\omega_{y\|x}) > b_{y|x} + a_y(1 - b_{y|x} - d_{y|\overline{x}}))$
$\wedge\ (b_x + a_x u_x > a_x)$

$$b^{\mathrm{III.B.2}}_{y\|x} \triangleq b^{\mathrm{I}}_{y\|x} - \frac{u_x(1-a_x)a_y(d^{\mathrm{I}}_{y\|x} - d_{y|\overline{x}})}{(d_x+(1-a_x)u_x)(1-a_y)}$$

$$d^{\mathrm{III.B.2}}_{y\|x} \triangleq d^{\mathrm{I}}_{y\|x} - \frac{u_x(1-a_x)(1-a_y)(d^{\mathrm{I}}_{y\|x} - d_{y|\overline{x}})}{(d_x+(1-a_x)u_x)(1-a_y)}$$

$$u^{\mathrm{III.B.2}}_{y\|x} \triangleq u^{\mathrm{I}}_{y\|x} + \frac{u_x(1-a_x)(d^{\mathrm{I}}_{y\|x} - d_{y|\overline{x}})}{(d_x+(1-a_x)u_x)(1-a_y)}$$

$$a^{\mathrm{III.B.2}}_{y\|x} \triangleq a_y \ .$$

where $\mathrm{E}(\omega_{y\|x}) = b^{\mathrm{I}}_{y\|x} + a^{\mathrm{I}}_{y\|x}u^{\mathrm{I}}_{y\|x}$.

Then $\omega_{y\|x}$ is called the conditional inference opinion of ω_x by $\omega_{y|x}$ and $\omega_{y|\overline{x}}$. The opinion $\omega_{y\|x}$ expresses the belief in y being true as a function of the beliefs in x and the two sub-conditionals $y|x$ and $y|\overline{x}$. The conditional inference operator is a ternary operator, and by using the function symbol '$\circledcirc$' to designate this operator, we define $\omega_{y\|x} \triangleq \omega_x \circledcirc (\omega_{y|x}, \omega_{y|\overline{x}})$.

Justification

The expressions for conditional inference seem quite complex, and the best justification can be found in its geometrical interpretation in the triangle of Fig.1.

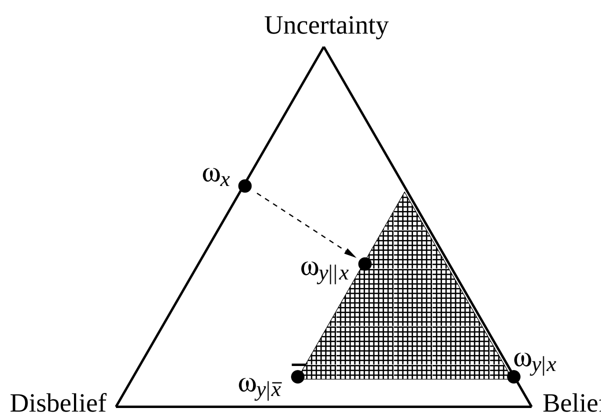

Figure 2: Example conditional inference sub-triangle

The opinions of the two sub-conditionals define a sub-triangle within which the opinion of the consequent must be located. In Figure 2 the opinions of the sub-conditionals are for example $\omega_{y|x} = (0.90,\ 0.02,\ 0.08,\ 0.50)$ and $\omega_{y|\overline{x}} = (0.40,\ 0.52,\ 0.08,\ 0.50)$, and the sub-triangle they define appears shaded. Now let for example the opinion about the antecedent be $\omega_x = (0.00,\ 0.38,\ 0.62,\ 0.50)$. The opinion

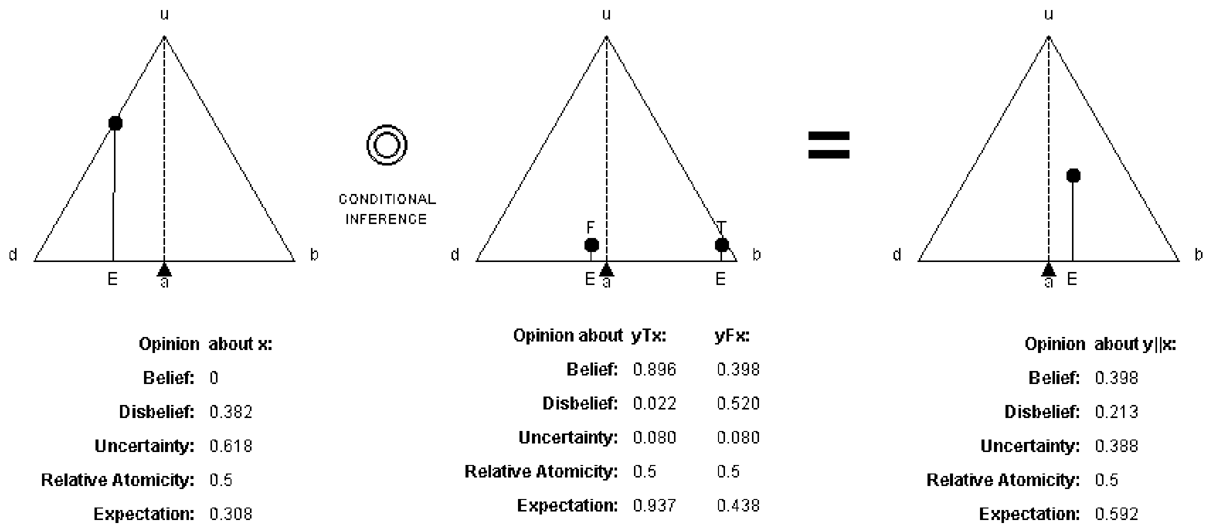

Figure 3: Example from online demo of the conditional inference operator[2]

of the consequent $\omega_{y\|x} = (0.40,\ 0.21,\ 0.39,\ 0.50)$ can then be obtained by mapping the position of the antecedent ω_x in the main triangle onto a position that relatively seen has the same belief, disbelief and uncertainty components in the sub-triangle.

Figure 3 shows the same example using the online demo of subjective logic [2]. Here the opinion about the antecedent be ω_x is shown in the leftmost triangle. The opinions about the two sub-conditionals $\omega_{y|x}$ and $\omega_{y|\overline{x}}$ are shown in the middle triangle, and the opinion about the consequent $\omega_{y\|x}$ is shown in the rightmost triangle.

This example is particularly simple because the sub-triangle is equal-sided. This is due to the fact that the two sub-conditionals have equal uncertainty component. It is however perfectly possible to let the two sub-conditional opinions have different uncertainty components in which case the sub-triangle no longer will be equal-sided. It is also possible to let the opinion about the sub-conditional $y|\overline{x}$ appear to the right of the opinion about the sub-conditional $y|x$ in which case the sub-triangle will be flipped around so that the mapping from the antecedent to the consequent goes to the opposite side of the sub-triangle. It is also possible that the sub-triangle is reduced to a line when the angle of the line between the two sub-conditional opinion points is steeper than the left and right sides of the triangle. The 9 different cases in Def.2 cover the various geometrical possibilities. It would take too long to provide an explanation for all cases, and we will explain Case I which is when the sub-triangle is reduced to a line.

- Case I:
  We know from probability calculus that

$$p(y) = p(y|x)p(x) + p(y|\overline{x})p(\overline{x}) \ . \tag{23}$$

We require the conditional inference operator to be compatible with this formula so that we can write

$$\mathrm{E}(\omega_{y\|x}) = \mathrm{E}(\omega_{y|x})\mathrm{E}(\omega_x) + \mathrm{E}(\omega_{y|\overline{x}})\mathrm{E}(\omega_{\overline{x}}) \ . \tag{24}$$

The relative atomicity of y is assumed to be a constant, so that the following holds:

$$\begin{aligned} a_y &= a_{y\|x} \\ &= a_{y|x} \\ &= a_{y|\overline{x}} \ . \end{aligned} \tag{25}$$

It is then possible to express $\mathrm{E}(\omega_{y\|x})$ on the form:

$$\mathrm{E}(\omega_{y\|x}) = b_{y\|x} + a_{y\|x}u_{y\|x} \ . \tag{26}$$

By substituting the expressions for the probability expectation values in Eq.(26) the expressions for $b_{y\|x}$ and $u_{y\|x}$ emerge. The expression for $d_{y\|x}$ emerges by solving $b_{y\|x} + d_{y\|x} + u_{y\|x} = 1$.

The best way to be convinced by the soundness of Def.2 is simply to try the online demo at [2].

5 Example: Michael's Umbrella

This section describes a simple scenario that illustrates how the conditional inference operator can be applied.

A simple man, named Michael, lives six months of the year in England and the other six months in Australia. Michael experiences very different weather in each location and this is reflected in the way he uses his umbrella.

5.1 Michael in England

In England, it often rains, and when it does not rain the sun is never too hot. If it rains, then Michael usually carries an umbrella, and if it does not rain he usually does not carry an umbrella. We define the propositions:

x	: 'It rains'
y	: 'Michael carries an umbrella'
$y\|x$	: 'IF it rains THEN Michael carries an umbrella'
$y\|\overline{x}$	: 'IF it does NOT rain THEN Michael carries an umbrella'
$y\|\|x$	: 'Michael carries an umbrella given the opinion about whether it rains'

Michael's English friend Edward has the following opinions about the sub-conditionals:

$$\begin{aligned} \omega_{y|x}^{\text{Edward}} &= (0.9,\ 0.0,\ 0.1,\ \tfrac{1}{2}) \\ \omega_{y|\overline{x}}^{\text{Edward}} &= (0.0,\ 0.9,\ 0.1,\ \tfrac{1}{2}) \end{aligned} \tag{27}$$

If one day Edward observes that it is not raining, so that $\omega_x^{\text{Edward}} = (0,\ 1,\ 0,\ \frac{1}{2})$, then Edward can infer that Michael probably does not carry an umbrella, expressed by $\omega_{y\|x}^{\text{Edward}} = (0.0,\ 0.9,\ 0.1,\ \frac{1}{2})$. Indeed, by looking out the window as Michael walks to the bus stop, Edward sees his friend without umbrella.

Assume that statistical data from the weather bureau in England indicates that it rains 50% of the time in England, and that Edward trusts this data to be correct. Edward can use this to estimate the likelihood of Michael carrying an umbrella at any given time. The opinion that it rains can thus be set to $\omega_x^{\text{Edward}} = (0.5,\ 0.5,\ 0,\ \frac{1}{2})$, so that the opinion that Michael carries an umbrella any particular day when no other weather info can be obtained is $\omega_{y\|x}^{\text{Edward}} = (0.45,\ 0.45,\ 0.10,\ \frac{1}{2})$.

5.2 Michael in Australia

In Australia it sometimes rains, and when it does not rain the sun can be very hot. If it rains, then Michael usually carries an umbrella. When it does not rain he sometimes carries an umbrella to protect his skin from the sun, but not always, in fact it is completely uncertain whether he carries an umbrella when it does not rain. We use the same propositions as in the English example.

Michael's Australian friend Andrew has the following opinions about the sub-conditionals:

$$\begin{aligned} \omega_{y|x}^{\text{Andrew}} &= (0.9,\ 0.0,\ 0.1,\ \tfrac{1}{2}) \\ \omega_{y|\overline{x}}^{\text{Andrew}} &= (0.0,\ 0.0,\ 1.0,\ \tfrac{1}{2}) \end{aligned} \tag{28}$$

If one day Andrew observes that it is not raining, so that $\omega_x^{\text{Andrew}} = (0,\ 1,\ 0,\ \frac{1}{2})$, what can Andrew infer from this? He applies the conditional inference function, and concludes that he is completely uncertain as to whether Michael carries an umbrella or not, expressed by $\omega_{y\|x}^{\text{Andrew}} = (0.0,\ 0.0,\ 1.0,\ \frac{1}{2})$. This type of conclusion cannot be inferred neither in binary logic nor in probability calculus.

Assume that statistical data from the weather bureau in Australia indicates that it rains 5% of the time in Australia, and that Andrew trusts this data to be correct. Andrew can use this to estimate the likelihood of Michael carrying an umbrella at any given time. The opinion that it rains can thus be set to $\omega_x^{\text{Andrew}} = (0.05,\ 0.95,\ 0,\ \frac{1}{2})$, so that the opinion that Michael carries an umbrella any particular day when no other weather info can be obtained is $\omega_{y\|x}^{\text{Andrew}} = (0.045,\ 0.000,\ 0.955,\ \frac{1}{2})$.

6 Discussion

Conditional inference demands some sort of necessary connection between antecedent x and the consequent y, which for example material (or indicative) implication ignores. Material implication denoted as $x \rightarrow y$ has the same truth table as, and is therefore logically equivalent to $\overline{x} \vee y$.

The idea of having a causal connection between the antecedent and the consequent can be traced back to Ramsey [7] who articulated what has become known as Ramsey's Test: *To decide whether you believe a conditional, provisionally or hypothetically add the antecedent to your stock of beliefs, and consider whether to believe the consequent*. By introducing Ramsey's test there has been a switch from truth and truth-functions to belief and whether to believe which can also be expressed in terms of probability and conditional probability. This idea was articulated by Stalnaker [9] and expressed by the so-called Stalnaker's Hypothesis as: $p(\text{IF } x \text{ THEN } y) = p(y|x)$. Stalnaker's Hypothesis is for example not consistent with the truth-functional interpretation of conditionals. For example when considering a standard pack of 52 playing cards we have $p(\text{king}|\text{ace}) \neq p((\text{NOT ace}) \vee \text{king})$ because $p(\text{king}|\text{ace}) = 0$ whereas $p((\text{NOT ace}) \vee \text{king}) = p(\text{NOT ace}) = \frac{12}{13}$.

According to Bayes rule the conditional probability of $y|x$ is equal to the probability of $x \wedge y$ divided by the probability of x, provided that the latter is not zero. Stalnaker's Hypothesis thus equates the probability of conditionals with classic conditional probability.

However, Lewis [5] argues that conditionals do not have truth-values and that they do not express propositions. In mathematical terms this means that given any propositions x and y, there is no proposition z for which $p(z) = p(y|x)$, so the conditional probability can not be the same as the probability of conditionals. Without going into detail we believe in Stalnaker's Hypothesis, and would argue against Lewis by simply saying that "IF x THEN Y" is equivalent to "$y|x$", and that this expresses a sub-conditional proposition with a truth value defined in case x is true, and undefined in case x is false.

Our approach is similar to that of conditional event algebras[3] where the set of events e.g. x, y in the probability space is augmented to include so-called class conditional events denoted by $y|x$. The primary objective in doing this is to define the conditional events in such a way that $p((y|x)) = p(y|x)$, that is so that the probability of the conditional event $y|x$ agrees with the conditional probability of y given x. There are a number of established conditional event algebras, each with their own advantages and disadvantages. In particular, one approach[1] used to construct them has been to employ a ternary truth system with values true, false and undefined, which corresponds well with the belief, disbelief and uncertainty components of opinions.

One explanation why conditionals have caused so much confusion is that most researchers have tried to describe causal relationships by using conditionals with one-dimensional truth. Our interpretation and explanation of conditional expressions is that they need two sub-conditionals to form a complete causal relationship.

MP is a sub-case of our definitions. Conditional inference in subjective logic is equivalent to MP in case for example $\omega_{y|x} = \omega_x = (1.0,\ 0.0,\ 0.0,\ 0.5)$ which is equivalent to $y|x$ and x being both TRUE. It can then be inferred that $\omega_y = (1.0,\ 0.0,\ 0.0,\ 0.5)$ which is equivalent to y being TRUE. In this regard it is worth mentioning McGee [6] who argues that counter-examples to MP exist. We argue that those arguments are no longer valid for the generalised MP described here because it allows the possibility of conditionals to be wrong by allocating an arbitrary amount of uncertainty to their truth value.

Probabilistic conditional inference is also a sub-case of our definitions. In probability calculus the conditionally inferred probability of y can be expressed according to Eq.(23). Conditional inference in subjective logic is equivalent to probabilistic conditional inference when $\omega_{y|x}$, $\omega_{y|\overline{x}}$ and ω_x are all dogmatic, in which case ω_x will also be dogmatic, and Eq.(23) can be applied directly.

By using the mapping between opinions and beta pdfs described by Eq.(22) it is also possible to base conditional inference on antecedents and conditionals expressed in the form of beta pdfs, which further contributes to making our approach more general.

7 Conclusion

The conditional inference operator described here represents a generalisation of the binary logic Modus Ponens rule and of probabilistic conditional inference. The advantage of our approach is that it is possible to take uncertainty and ignorance about the antecedent and the conditionals into account when analysing conditional inference and see the effect it has on the result. We see this work as a step forward in order to provide Shaferian belief theory with much needed operators. It also provides a bridge between belief theory on the one hand and binary logic and probability calculus on the other.

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